

FRACTIONAL PARTS OF POLYNOMIALS OVER THE PRIMES

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ABSTRACT. Let f be a polynomial of degree $k > 1$ with irrational leading coefficient. We obtain results of the form

$$\|f(p)\| < p^{-\sigma}$$

for infinitely many primes p that supersede those of Harman (1981, 1983) and Wong (1997).

1. INTRODUCTION

For $k \geq 2$, let ρ_k denote the supremum of positive numbers ν for which

$$\|\alpha p^k + \beta\| < p^{-\nu}$$

has infinitely many solutions in primes p for every irrational α and real β . Let σ_k denote the supremum of positive numbers ν for which

$$(1.1) \quad \|f_k(p)\| < p^{-\nu}$$

has infinitely many solutions in primes p whenever f_k is a polynomial of degree k with irrational leading coefficient. (See Matomaki [14] for the case $k = 1$, which presents different features from $k = 2, 3, \dots$.)

To state our main result we define the integer $J = J(f_k)$ as follows. For

$$\begin{aligned} f_k(x) &= \alpha x^k + \beta, \\ J(f_k) &= 2^{k+1} \quad (k \leq 5), \quad J(f_k) = k(k-1) \quad (k \geq 6) \end{aligned}$$

For other polynomials of degree k , let

$$J(f_k) = 2^{k-1} \quad (k \leq 7), \quad J(f_k) = 2k(k-1) \quad (k \geq 8).$$

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Theorem 1. *The inequality (1.1) has infinitely many solutions for*

$$\nu < \frac{2}{13} \quad (k = 2)$$

$$\nu < \frac{1}{10} \quad (k = 3)$$

$$\nu < \frac{0.4079}{J(f_k)} \quad (k \geq 4)$$

A few remarks are in order. Our first few results are of the form $\sigma_2 \geq 2/13$, $\sigma_3 \geq 1/10$, $\sigma_4 \geq 0.0509875$. Harman [11] obtained $\rho_2 \geq 2/13$, so we do not have a new result for the polynomials $\alpha x^2 + \beta$. Wong [16] obtained lower bounds for ρ_3, \dots, ρ_{11} with $\rho_3 \geq 5/56 = 0.0892\dots$ and $\rho_4 \geq 1/21 = 0.0476\dots$. In [9] Harman shows that $\sigma_k \geq 1/2^{2k-1}$, so that $\sigma_2 \geq 1/8$ and $\sigma_3 \geq 1/32$. Harman [10] gives improvements for $\sigma_4, \sigma_5, \dots$ including $\sigma_4 \geq 4/391 = 0.0102\dots$. Asymptotically, Harman [10] shows that

$$\sigma_k \geq \frac{1 + o(1)}{12k^2 \log k}.$$

Our improvements depend on obtaining new ‘arithmetical information’ to use in the Harman sieve [11, 12]. This amounts to giving upper bounds for trilinear exponential sums, of the form

$$(1.2) \quad \sum_{\ell \leq L} c_\ell \sum_{\substack{X < x \leq 2X \\ N/2 < xy < N}} a_x \sum_{Y < y \leq 2Y} b_y e(\ell g(mn)) \ll N^{1-\eta},$$

where $L = N^{\rho-\varepsilon/3}$ and either $|a_x| \leq 1$, $|b_y| \leq 1$ (Type II sums), or $|a_x| \leq 1$ and $b_y = 1$ identically (Type I sums). The point is to get the estimate over wider ranges than can be found in Baker and Harman [4], the present ‘state of the art’ for monomials. Here g is obtained from f_k by replacing its leading coefficient α_k by a/q , a convergent to α_k . Several devices come into play. We give a sharper bound in an auxiliary result on the number of solutions $y \in (Y, 2Y]$ of

$$\left\| \frac{say^3}{q} \right\| < \frac{1}{Z}$$

for a given integer $s < q$ by slightly adapting a result of Hooley [13]. For $k \geq 3$, we give a relatively simple argument that improves the lower bound on Y in (1.2) from $Y \gg L^2 N^{2\eta}$ (essentially) to $Y \gg L N^{2\eta}$, in the ‘Type II’ case. For $k \geq 6$, we use stronger results on simultaneous approximation to the coefficients of a large Weyl sum [3] than those

available to the authors of [4]; these simultaneous approximation results depend on the work of Bourgain, Demeter, and Guth [6].

We end this section with remarks on notation. We write ' $y \sim Y$ ' for ' $Y \leq y < 2Y$ '. We write $\langle s_1, \dots, s_k \rangle$, $[s_1, \dots, s_k]$ for greatest common divisor and least common multiple. Let $e(\theta) = e^{2\pi i \theta}$ and $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. Constants implied by 'O' and ' \ll ' depend only on k, ε . We suppose that the positive number ε is sufficiently small and let $\eta = \varepsilon/C_1(k)$ where $C_1(k)$ is a suitable large positive constant. Let a/q be a convergent (with q sufficiently large) to the continued fraction of α_k , where $f(x) = \alpha_k x^k + \dots + \alpha_1 x + \alpha_0$, and fix N with

$$(L_1 N^k)^{1/2} \ll q \ll (L_1 N^k)^{1/2},$$

where L_1 denotes $2N^{\rho-\varepsilon/2}$ with ρ defined by

$$\rho = \frac{2}{13} \ (k=2), \ \rho = \frac{1}{10} \ (k=3), \ \rho = \frac{0.4079}{J(f)} \ (k \geq 4).$$

Clearly (much as in [4]) it suffices to prove that there is a positive number of primes in the set

$$A = \left\{ \frac{N}{2} < n \leq N : \|g(n)\| < L_1^{-1} \right\}.$$

2. SMALL VALUES OF A MONOMIAL (mod q).

Let $1 \leq Y < q$, $1 \leq D < q$, $Z \geq 2$, $1 \leq s < q$. For later use we need to bound the number of solutions of the inequality

$$(2.1) \quad \left\| \frac{say^k}{q} \right\| < \frac{1}{Z}$$

for which $y \in (Y, 2Y]$ and $\langle y, q \rangle \leq D$. Denote this number by $\mathcal{N}_k(Y, D, Z, s)$.

Lemma 1. (i) *With the above notations, we have*

$$\mathcal{N}_k(Y, 1, Z, s) \ll q^{1+\eta} Z^{-1}.$$

(ii) *Whenever $sD^k < q$, we have*

$$\mathcal{N}_k(Y, D, Z, s) \ll q^{1+2\eta} Z^{-1}.$$

Proof. For part (i) see [4, Lemma 6]. To deduce part (ii) it suffices to show that for $d \mid q$ the number of $y \sim Y$ with (2.1) and $\langle y, q \rangle = d \leq D$ is

$$\ll q^{1+\eta} Z^{-1}.$$

Write $y = y_1 d$, $q = q_1 d$, $(y_1, q_1) = 1$. Then (2.1) implies

$$\left\| \frac{sd^{k-1}y_1^k}{q_1} \right\| < Z^{-1}$$

and $y_1 \sim \frac{Y}{d} < q_1$. Since $sd^{k-1} < q_1$, the desired bound follows from part (i). \square

Lemma 2. *Let Y, Z be positive numbers in $[1, N^3]$. Then*

$$\begin{aligned} \mathcal{N}_3(Y, q, Z, s) \ll Y^{1/2} + N^\eta \left(YZ^{-1/4} + Y \left(\frac{\langle s, q \rangle}{q} \right)^{1/4} \right. \\ \left. + Y^{1/4} q^{1/4} Z^{-1/4} \right). \end{aligned}$$

Proof. In the case $\langle s, q \rangle = 1$, this follows from Hooley [13, Theorem 1]. For the general case, we rewrite (2.1) as

$$\left\| \frac{s_1 a y^3}{q_1} \right\| < \frac{1}{Z}$$

where $d = (s, q)$, $s = s_1 d$, $q = q_1 d$, $(s_1, q_1) = 1$. \square

Lemma 3. (i) *Let s, Y be positive integers less than q and let $Z \geq 2$. Then*

$$\mathcal{N}(Y, 1, Z, s) \ll q^\eta \left(\frac{Y + q^{1/2}}{Z^{1/2}} \right).$$

(ii) *Let $D \geq 1$. Whenever $sD^2 < q$, in addition to the above hypotheses, we have*

$$\mathcal{N}(Y, D, Z, s) \ll q^{2\eta} \left(\frac{Y + q^{1/2}}{Z^{1/2}} \right).$$

Proof. For (i), see [4, Lemma 9]. We deduce (ii) from (i) by an argument used in proving Lemma 1. \square

Our next task is to ‘allow s to vary’ in the counting performed in Lemmas 1–3. For $S_0 \geq 1$, $S_1 \geq 1$ it is convenient to write

$$\begin{aligned} \mathcal{A}(S_0, S_1, d_0, d_1) \\ = \{s = s_0 s_1 : \langle s_0, s_1 \rangle = 1, s_0 \text{ square-full}, \\ s_1 \text{ square-free}, d_0 \mid s_0, d_1 \mid s_1\} \end{aligned}$$

whenever d_0, d_1 are positive integers.

Lemma 4. *For $S_0 \leq N$, $S_1 \leq N$,*

$$\#\mathcal{A}(S_0, S_1, d_0, d_1) \ll N^\eta S_0^{1/2} S_1 d_0^{-1/2} d_1^{-1}.$$

Proof. The number of possible s_1 here is $\ll S_1 d_1^{-1}$. Any s_0 occurring can be written

$$(2.2) \quad s_0 = d_0 uv$$

where $p \mid u$ implies $p \mid d_0$, and v is squarefull and relatively prime to d_0 .

Obviously, $v \leq 2S_0/d_0$, so there are $O(S_0^{1/2} d_0^{-1/2})$ possible v .

It remains to show that H , the number of different u that can occur in (2.2), is $O(N^\eta)$.

Now, writing $p_1 < \dots < p_t$ for the prime divisors of d_0 , we find that H is at most equal to the number of tuples (m_1, \dots, m_t) , m_i a non-negative integer, with

$$(2.3) \quad m_1 \log p_1 + \dots + m_t \log p_t \leq 2S_0.$$

A little thought (replace p_1, \dots, p_t in (2.3) by the first t primes q_1, \dots, q_t) shows that

$$H \leq \Psi(2S_0, q_t),$$

in the usual notation for smooth numbers. Since $q_t < (1 + \varepsilon)t \log t$ if t is large, and $t < (1 + \varepsilon) \log N / \log \log N$, we have

$$H \leq \Psi(2N, 2 \log N).$$

An appeal to Theorem 1 of de Bruijn [8] now yields

$$H \ll N^\eta,$$

and the lemma follows. \square

Let

$$\mathcal{M}_k(Y, Z, S_0, S_1) = \# \left\{ y \sim Y : (y, q) \leq N^\rho, \left\| \frac{say^k}{q} \right\| < \frac{1}{Z} \right. \\ \left. \text{for some } s \in \mathcal{A}(S_0, S_1, 1, 1) \right\}.$$

Summing over s in Lemma 1 and Lemma 3, we obtain

$$(2.4) \quad \mathcal{M}_k(Y, Z, S_0, S_1) \ll N^{3\eta} S_0^{1/2} S_1 \frac{q}{Z}$$

whenever $1 \leq Y < q$, $Z \geq 2$ and

$$(2.5) \quad 4S_0 S_1 N^{k\rho} < q,$$

while under the same conditions on Y , Z , S_0 , S_1 ,

$$(2.6) \quad \mathcal{M}_2(Y, Z, S_0, S_1) \ll N^{3\eta} S_0^{1/2} S_1 (Y + q^{1/2}) Z^{-1/2}.$$

To obtain a bound for $\mathcal{M}_3(Y, Z, S_0, S_1)$, we restrict $s = s_0 s_1$ to values with $(s_0, q) = d_0$, $(s_1, q) = d_1$, at a cost of a factor $O(N^\eta)$. Now Lemma 2 and Lemma 4 together yield

$$(2.7) \quad \mathcal{M}_3(Y, Z, S_0, S_1) \ll N^{3\eta} S_0^{1/2} S_1 (Y^{1/2} + YZ^{-1/4} + Yq^{-1/4} + Y^{1/4} q^{1/4} Z^{-1/4})$$

(the factor $(d_0 d_1)^{1/4}$ in the third term in the bound in Lemma 2 is cancelled by the factor $d_0^{-1/2} d_1^{-1}$ in Lemma 4).

In our applications we shall always have (2.5). If we assume this additional condition for $k = 3$, there are no solutions of

$$\left\| \frac{say^3}{q} \right\| < \frac{1}{q} \quad \text{with } \langle y, q \rangle \leq N^\rho$$

counted in (2.7). So we may suppose that $Z < q$, and we obtain

$$(2.8) \quad \mathcal{M}_3(Y, Z, S_0, S_1) \ll N^{2\eta} S_0^{1/2} S_1 (Y^{1/2} + YZ^{-1/4} + Y^{1/4} q^{1/4} Z^{-1/4}).$$

3. TYPE I SUMS

Our most basic tool is obtained by combining Theorem 5.1 of [1] (with a correction in [2]) and Theorem 4 of [6].

Lemma 5. *Let f be a polynomial of degree k , $f(x) = \gamma_k x^k + \dots + \gamma_1 x + \gamma_0$. Let $M \geq 1$ and $X \geq 1$, with $M = 1$ when $J = J(f) \neq 2^{k-1}$. Suppose that for some subinterval I of $[\frac{X}{2}, X]$ we have*

$$\sum_{m=1}^M \left| \sum_{x \in I} e(mf(x)) \right| \geq P \geq MX^{1-1/J+\eta}.$$

Then there are natural numbers s and integers u_1, \dots, u_k with $\langle s, u_1, \dots, u_k \rangle = 1$; $\langle s, u_2, \dots, u_k \rangle \leq MX^\eta$, in the case $J(f) = 2^{k-1}$;

$$s \ll (MXP^{-1})^k X^\eta,$$

$$|s\gamma_j - u_j| \ll M^{-1} (MXP^{-1})^k X^{1-j} \quad (j = 1, 2, \dots, k).$$

Lemma 6. *Let $u \in \mathbb{Z}$, $d \in \mathbb{N}$, $B \geq 1$, $L \geq 1$. Let \mathcal{N} be the number of solutions of*

$$\ell u \equiv b \pmod{d} \quad (1 \leq \ell \leq L, 1 \leq b \leq B)$$

Then

$$\mathcal{N} \leq \min(L, B) + \frac{BL}{d}.$$

Proof. The congruence has no solution unless $\langle u, d \rangle \mid b$. For fixed b , the number of possibilities for $\ell \pmod{\frac{d}{\langle u, d \rangle}}$ is at most 1. Hence

$$\mathcal{N} \leq \frac{B}{\langle u, d \rangle} \left(\frac{L\langle u, d \rangle}{d} + 1 \right) \leq \frac{BL}{d} + B.$$

On the other hand, for given ℓ , the number of possible b is at most $\frac{B}{d} + 1$. This gives the alternative upper bound $\frac{BL}{d} + L$. \square

Lemma 7. *Let $k \geq 2$. Let $f(x) = \gamma_k x^k + \dots + \gamma_1 x$. Let $X \geq 1$, $1 \leq L \leq X$. Suppose there are integers s, u_1, \dots, u_k , $s \leq X$, $\langle s, u_1, \dots, u_k \rangle = 1$, and if $J(f) = 2^{k-1}$, $\langle s, u_2, \dots, u_k \rangle \ll LN^\eta$, such that*

$$(3.1) \quad |s\gamma_j - u_j| \leq (2k^2)^{-1} L^{-1} X^{1-j} \quad (1 \leq j \leq k).$$

Let

$$\beta_j = \gamma_j - \frac{u_j}{s}, \quad F(x) = \sum_{j=1}^k \beta_j x^j, \quad G(x) = \sum_{j=1}^k u_j x^j,$$

$$S(s, \ell G) = \sum_{v=1}^s e\left(\frac{\ell G(v)}{s}\right).$$

Then we have, for any subinterval I of $[\frac{X}{2}, X]$,

$$(3.2) \quad \sum_{\ell=1}^L \left| \sum_{n \in I} e(\ell f(x)) - s^{-1} S(s, \ell G) \int_I e(\ell F(z)) dz \right| \\ \ll \begin{cases} N^{2\eta} L s^{1-1/k} & \text{if } J(f) = 2^{k-1} \\ N^{2\eta} (L s^{1-1/k} + s) & \text{otherwise.} \end{cases}$$

Proof. Following the proof of [1, Lemma 4.4], we find that

$$\sum_{n \in I} e(\ell f(x)) - s^{-1} S(s, \ell G) \sum_{n \in I} e(\ell F(n)) \\ \ll s^{-1} \sum_{b=1}^{s-1} \left\| \frac{b}{s} \right\|^{-1} \left| \sum_{v=1}^s e\left(\frac{\ell G(v) + bv}{s}\right) \right|$$

and

$$\sum_{n \in I} e(\ell F(n)) = \int_I e(\ell F(z)) dz + O(1).$$

Moreover, by a standard estimate [7],

$$\begin{aligned} S(s, \ell G) &\ll \langle \ell, s \rangle^{1/k} s^{1-1/k} N^\eta, \\ \sum_{v=1}^s e\left(\frac{\ell G(v) + bv}{s}\right) &\ll \langle \ell u_1 + b, \ell u_2, \dots, \ell u_k, s \rangle^{1/k} s^{1-1/k} N^\eta \\ &\ll D_\ell s^{1-1/k} N^\eta \end{aligned}$$

where $D_\ell = \min(LN^\eta, \langle \ell u_1 + b, s \rangle)^{1/k}$ if $J(f) = 1$ and $D_\ell = \langle \ell u_1 + b, s \rangle$ otherwise.

It follows that

$$\begin{aligned} \sum_{n \in I} e(\ell f(x)) - s^{-1} S(s, \ell G) \int_I e(\ell F(z)) dz \\ \ll \langle \ell, s \rangle^{1/k} s^{1-1/k} N^\eta + N^\eta s^{-1/k} \sum_{b=1}^{s-1} \left\| \frac{b}{s} \right\|^{-1} D_\ell. \end{aligned}$$

We now sum the absolute values of the left-hand side over $\ell \leq L$. Since the contribution from $\sum_{\ell=1}^L \langle \ell, s \rangle^{1/k}$ is $\ll LN^\eta$, a splitting-up argument shows that we need only prove the bound

$$\ll \begin{cases} N^\eta L s^{1-1/k} & (J(f) = 2^{k-1}) \\ N^\eta (L s^{1-1/k} + s) & (\text{otherwise}) \end{cases}$$

for the quantity

$$(3.3) \quad \frac{s^{1-1/k}}{B} \sum_{\frac{B}{2} \leq b < 2B} \sum_{\substack{\ell=1 \\ d | \ell u + b}}^L A_d$$

where $d \mid s$ and

$$A_d = \begin{cases} \min(L^{1/k} N^\eta, d^{1/k}) & \text{if } J(f) = 1 \\ d^{1/k} & \text{otherwise.} \end{cases}$$

Applying Lemma 8, the left-hand side of (3.3) is

$$\ll \frac{s^{1-1/k}}{B} \min(L^{1/k} N^\eta, d^{1/k}) \left(\frac{BL}{d} + B \right) \ll N^\eta s^{1-1/k} L$$

if $J(f) = 1$. Otherwise we obtain

$$\ll \frac{s^{1-1/k}}{B} d^{1/k} \left(\frac{BL}{d} + B \right) \ll s^{1-1/k} L + s.$$

This proves the lemma. \square

Lemma 8. *Let $k \geq 2$ and $Y \ll N^{1-5\rho/2}$ ($k = 2$), $Y \ll N^{1/2+\rho}$ ($k \geq 3$). Then, with g as defined in Section 1, we have*

$$(3.4) \quad \sum_{\ell=1}^L \sum_{y \sim Y} \left| \sum_{n \in I(y)} e(\ell g(y n)) \right| \ll N^{1-2\eta},$$

where $I(y) = \left(\frac{N}{2y}, \frac{N}{y} \right]$.

Proof. Let \mathcal{S} be the set of $y \sim Y$ with $\langle y, q \rangle \leq N^\rho$ and

$$(3.5) \quad \sum_{\ell=1}^L \left| \sum_{n \in I(y)} e(\ell g(y n)) \right| > N^{1-2\eta} Y^{-1}.$$

It suffices to show that

$$T := \sum_{y \in \mathcal{S}} \sum_{\ell=1}^L \left| \sum_{n \in I(y)} e(\ell g(y n)) \right| \ll N^{1-2\eta}.$$

To see this, the contribution in (3.4) from $y \sim Y$ for which (3.5) fails is $< N^{1-2\eta}$. The contribution from $y \sim Y$ for which $(\langle y, q \rangle)$ is a fixed divisor d of q , $d > N^\rho$, is

$$\ll \frac{Y}{d} L \frac{N}{Y} \ll N^{1-3\eta}$$

and our claim follows on summing over d .

Given $y \in \mathcal{S}$, we apply (3.5) in Lemmas 5 and 7. Here $\gamma_k = ay^k/q$, $\gamma_j = \alpha_j y^j$ ($j < k$). Suppose first that $J(f) = 2^{k-1}$. Take $X = \frac{N}{Y}$, $M = L$ in Lemma 5. Then

$$P = N^{1-2\eta} Y^{-1} \geq L \left(\frac{N}{Y} \right)^{1-1/J+\eta}$$

since $Y \ll N^{1-J\rho}$. The integers s, u_1, \dots, u_k provided by Lemma 5 satisfy (3.1), since $k \leq J$ and so

$$L^k N^\eta \ll \frac{N}{Y} N^{-\eta}.$$

Lemma 7 yields

$$(3.6) \quad \begin{aligned} \frac{N^{1-2\eta}}{Y} &\ll \sum_{\ell=1}^L \left| \sum_{n \in I(y)} e(\ell g(n)) \right| \\ &\ll \sum_{\ell=1}^L \left| s^{-1} S(s, \ell G) \int_{I(y)} e(\ell F(z)) dz \right| + L^k N^{3\eta k} \end{aligned}$$

where $I(y) = \left(\frac{N}{2y}, \frac{N}{y}\right]$. Here we suppress dependence of F, G on y .

The last term is of smaller order than $\frac{N^{1-2\eta}}{Y}$ in (3.6), so that

$$(3.7) \quad \frac{N^{1-2\eta}}{Y} \ll \sum_{\ell=1}^L \left| s^{-1} S(s, \ell G) \int_{I(y)} e(\ell F(z)) dz \right|.$$

We now show that that (3.7) also holds when $J(f) \neq 2^{k-1}$. Select $m_0 = m_0(y)$ such that

$$\left| \sum_{n \in I(y)} e(m_0 g(n)) \right| \geq P := \frac{N^{1-2\eta}}{YL}.$$

We have $Y \ll N^{1-J\rho}$, as is easily verified. Hence

$$P \geq \left(\frac{N}{Y}\right)^{1-\frac{1}{J}+\eta}.$$

We apply Lemma 5 with $f = m_0 g$, obtaining integers s', u'_1, \dots, u'_k with $s' \ll L^k N^{3k\eta}$,

$$|s' m_0 \gamma_j - u'_j| \ll L^k N^{3k\eta} \left(\frac{N}{Y}\right)^{-j} \quad (j = 1, \dots, k).$$

Let $d = \langle s' m_0, u'_1, \dots, u'_k \rangle$ and $s = \frac{s' m_0}{d}$, $u_j = \frac{u'_j}{d}$. Then

$$|s \gamma_j - u_j| \ll L^k N^{3\rho k} \left(\frac{N}{Y}\right)^{-j} \ll L^{-1} N^{-\eta} \left(\frac{N}{Y}\right)^{-(j-1)}$$

since $J \geq k+1$. Thus we can apply Lemma 7. In the analogue of (3.6), the second term on the right-hand side is now

$$\ll (L s^{1-1/k} + s) N^\eta \ll N^{(k+1)\rho} \ll \frac{N}{Y} N^{-3\eta},$$

and we again end up with (3.7).

Factorizing s as $s = s_0 s_1$ with s_0 square-full, s_1 square-free, and $\langle s_0, s_1 \rangle = 1$, we have

$$(3.8) \quad s^{-1} S(s, \ell G) \ll \left(\frac{s_0}{\langle s_0, \ell \rangle}\right)^{-1/k} \left(\frac{s_1}{\langle s_1, \ell \rangle}\right)^{-1/2}.$$

See Cochrane [7] for more general results. The estimate

$$(3.9) \quad \int_{I(y)} e(\ell F(z)) dz \ll \min \left(\frac{N}{Y}, \ell^{-1/k} \left| \frac{y^k a}{q} - \frac{u_1}{s} \right|^{-1/k} \right)$$

is a consequence of Vaughan [15, Theorem 7.3]. Putting the trivial estimate in (3.9) together with (3.7), (3.8), we have

$$\frac{N^{1-2\eta}}{Y} \ll \sum_{\ell=1}^L \langle s, \ell \rangle^{1/2} \cdot \frac{1}{s_0^{1/k} s_1^{1/k}} \frac{N}{Y},$$

whence

$$(3.10) \quad s_0^{1/k} s_1^{1/2} \ll LN^{3\eta}.$$

We now subdivide \mathcal{S} into $O((\log N)^3)$ classes according to the values of $s_0 = s_0(y)$, $s_1 = s_1(y)$ and

$$\left| \frac{y^k a}{q} - \frac{u_k}{s_0 - s_1} \right|.$$

In each class $\mathcal{S}(Z, S_0, S_1)$, we have $s_0 \sim S_0$, $s_1 \sim S_1$ with

$$(3.11) \quad S_0^{1/k} S_1^{1/2} \ll LN^{3\eta},$$

and, with Z_0 defined below, $Z = 2^{-j} Z_0 \geq 2$, also

$$(3.12) \quad \frac{1}{2s_0 s_1 Z} \leq \left| \frac{y^k a}{q} - \frac{u_k}{s_0 s_1} \right| < \frac{1}{s_0 s_1 Z} \quad \text{or} \quad (\text{if } Z = Z_0) \left| \frac{y^k a}{q} - \frac{u_k}{s_0 s_1} \right| < \frac{1}{s_0 s_1 Z}.$$

Here

$$L^{-1/k} (Z_0 S_0 S_1)^{1/k} = \frac{N}{Y}.$$

From (3.7), (3.8), (3.9), (3.11), (3.12) there is a class $\mathcal{S}^* = \mathcal{S}(Z, S_0, S_1)$ and an $L_0 \in [1, L)$ such that

$$\begin{aligned} T &\ll N^\eta \sum_{y \in \mathcal{S}^*} \sum_{\ell \sim L_0} \langle s_0(y) s_1(y), \ell \rangle^{1/2} L_0^{-1/k} Z^{1/k} S_1^{-\frac{1}{2} + \frac{1}{k}} \\ &\ll S_1^{-\frac{1}{2} + \frac{1}{k}} L^{1-1/k} N^{2\eta} Z^{1/k} \# \mathcal{S}^*. \end{aligned}$$

We can estimate $\# \mathcal{S}^*$ using the results of Section 2, since for every $y \in \mathcal{S}^*$, there is an $s \in \mathcal{A}(S_0, S_1, 1, 1)$ with

$$\left\| \frac{s a y^k}{q} \right\| < \frac{1}{Z}.$$

Thus, in the notation of Section 2,

$$(3.13) \quad T \ll S_1^{-\frac{1}{2} + \frac{1}{k}} L^{1-1/k} N^{2\eta} Z^{1/k} \mathcal{M}_k(Y, Z, S_0, S_1).$$

We now conclude the proof by considering separately the cases $k = 2$, $k = 3$, and $k \geq 4$. It is easy to verify the condition (2.5) needed for our bounds on \mathcal{M}_k , since

$$S_0 S_1 N^{k\rho} \ll N^{2k\rho} \ll N \ll q N^{-\rho/2}$$

$k = 2$. Recalling (2.6), we deduce from (2.5) that

$$\begin{aligned} T &\ll L^{1/2} N^{3\eta} S_0^{1/2} S_1(Y + q^{1/2}) \\ &\ll L^{1/2} N^{3\eta} S_0^{1/2} S_1 N^{1-5\rho/2} \end{aligned}$$

since $\frac{1}{2} + \frac{\rho}{4} < 1 - \frac{5\rho}{2}$. Using (3.11),

$$T \ll L^{5/2} N^{1+9\eta-5\rho/2} \ll N^{1-2\eta}.$$

$k = 3$. Recalling (2.4), (2.8),

$$\begin{aligned} (3.14) \quad T &\ll L^{2/3} N^{4\eta} Z^{1/3} S_0^{1/2} S_1^{5/6} \min\left(\frac{q}{Z}, Y^{1/2} + \frac{(Y + Y^{1/4} q^{1/4})}{Z^{1/4}}\right) \\ &\ll L^{7/3} N^{9\eta} Z^{1/3} \min\left(\frac{q}{Z}, N^{1/4+\rho/2} + \frac{N^{1/2+\rho}}{Z^{1/4}}\right), \end{aligned}$$

since $Y^{1/4} q^{1/4} \ll N^{1/8+\rho/4+3/8+\rho/8}$. Next,

$$\begin{aligned} &L^{7/3} N^{9\eta} Z^{1/3} \min\left(\frac{q}{Z}, N^{1/4+\rho/2}\right) \\ &\leq L^{7/3} N^{9\eta} q^{1/3} N^{(1/4+\rho/2)2/3} \\ &\ll N^{2/3+\rho(7/3+1/6+1/3)} \ll N^{1-2\eta} \end{aligned}$$

and

$$\begin{aligned} &L^{7/3} N^{9\eta} Z^{1/3} \min\left(\frac{q}{Z}, \frac{N^{1/2+\rho}}{Z^{1/4}}\right) \\ &\leq L^{7/3} N^{9\eta} q^{1/9} (N^{1/2+\rho})^{8/9} \\ &\ll N^{11/18+\rho(7/3+1/18+8/9)} \ll N^{1-2\eta}, \end{aligned}$$

completing the discussion for $k = 3$.

$k \geq 4$. Using (2.4), (3.11),

$$\begin{aligned}
T &\ll S_1^{-1/2+1/k} L^{1-1/k} N^{5\eta} Z^{1/k} \min\left(Y, \frac{S_0^{1/2} S_1 q}{Z}\right) \\
&\ll L^{1-1/k} N^{5\eta} Y^{1-1/k} S_0^{1/2k} S_1^{-1/2+2/k} q^{1/k} \\
&\ll N^{\rho(1-1/k)+(1/2+\rho)(1-1/k)+\rho/2+1/2+\rho/2k} \\
&\ll N^{1-2\eta}
\end{aligned}$$

since (as we easily verify)

$$\rho\left(\frac{5}{2} - \frac{3}{2k}\right) < \frac{1}{2k}.$$

This completes the proof of Lemma 8. \square

4. TYPE II SUMS

Our object in the present section is to prove

Lemma 9. *For $k = 2$, let $N^{2\rho} \ll Y \ll N^{1-4\rho}$. For $k \geq 3$, let $N^\rho \ll Y \ll N^{1-2J\rho}$, $J = J(f_k)$. Let $|a_k| \leq 1$ ($x \leq \frac{N}{Y}$), $|b_y| \leq 1$ ($y \sim Y$). Then*

$$S := \sum_{\ell=1}^L \left| \sum_{\substack{x \leq \frac{N}{Y} \\ \frac{N}{2} < xy \leq N}} a_x \sum_{y \sim Y} b_y e(\ell g(xy)) \right| \ll N^{1-\eta}.$$

We observe that the condition $\frac{N}{2} < xy \leq N$ may be removed at the cost of a log factor [12, Section 3.2], and we shall show that S' , defined like S without this condition, is $\ll N^{1-2\eta}$.

Proof of Lemma 9 for $k \geq 3$. We may write

$$S' = \sum_{\ell=1}^L c_\ell \sum_{y \sim Y} b_y \sum_{x \leq \frac{N}{Y}} a_x e(\ell g(xy))$$

where $|c_\ell| = 1$, so that

$$|S'| \leq \sum_{x \leq \frac{N}{Y}} \left| \sum_{\ell=1}^L c_\ell \sum_{y \sim Y} b_y e(\ell g(xy)) \right|.$$

By the Cauchy-Schwarz inequality,

(4.1)

$$\begin{aligned} |S'|^2 &\leq \frac{N}{Y} \sum_{x \leq \frac{N}{Y}} \left| \sum_{\ell=1}^L \sum_{y \sim Y} c_\ell b_y e(\ell g(xy)) \right|^2 \\ &= \frac{N}{Y} \sum_{\ell_1, \ell_2=1}^L \sum_{y_1, y_2 \sim Y} c_{\ell_1} \bar{c}_{\ell_2} b_{y_1} \bar{b}_{y_2} \sum_{x \leq \frac{N}{Y}} e(\ell_1 g(xy_1) - \ell_2 g(xy_2)). \end{aligned}$$

The contribution from quadruples $(\ell_1, \ell_2, y_1, y_2)$ with $\ell_1 y_1^k = \ell_2 y_2^k$ is

$$\ll \left(\frac{N}{Y} \right)^2 N^\eta LY \ll N^{2-4\eta}$$

by a divisor argument, since $Y \geq N^\rho$. Hence it suffices to show that

$$\left| \sum_{x \leq \frac{N}{Y}} e(\ell_1 g(xy_1) - \ell_2 g(xy_2)) \right| < \frac{N^{1-4\eta}}{Y} L^{-2}$$

for a given quadruple with $\ell_1 y_1^k \neq \ell_2 y_2^k$, $\ell_j \sim L$, $y_j \sim Y$.

Suppose the contrary. We may apply Lemma 5 with $X = \frac{N}{Y}$, $M = 1$, and $P = \frac{N^{1-4\eta}}{Y} L^{-2}$. We have

$$P \geq X^{1-\frac{1}{J}+\eta}$$

since

$$X^{1-\frac{1}{J}+\eta} P^{-1} \leq N^{5\eta} \left(\frac{N}{Y} \right)^{-\frac{1}{J}} L^2 \leq N^{5\eta} (N^{2J\rho})^{-\frac{1}{J}} L^2 \leq 1.$$

Hence there exists a natural number s and an integer u ,

$$\begin{aligned} s &\ll (N^{2\eta} L^2)^k \ll N^{2k\rho-\eta}, \\ \left| s \left(\frac{\ell_1 a y_1^k - \ell_2 a y_2^k}{q} \right) - u \right| &< N^{2k\rho} \left(\frac{N}{Y} \right)^{-k}, \end{aligned}$$

that is,

$$(4.2) \quad \left\| \frac{sa}{q} (\ell_1 y_1^k - \ell_2 y_2^k) \right\| < N^{2k\rho} \left(\frac{N}{Y} \right)^{-k}.$$

The right-hand side of (4.2) is less than $1/q$, since $Y \ll N^{1/5}$ and

$$(4.3) \quad N^{2k\rho} \left(\frac{N}{Y} \right)^{-k} q \ll N^{(2k+\frac{1}{2})\rho-3k/10} \ll N^{-\eta}$$

(it is easy to verify that $\rho < \frac{3k}{20k+1}$). However, the integer $sa(\ell_1 y_1^k - \ell_2 y_2^k)$ is not divisible by q , since

$$1 \leq |s(\ell_1 y_1^k - \ell_2 y_2^k)| \ll N^{(2k+1)\rho} Y^k \ll q N^{-\eta}$$

by the same inequality

$$\left(2k + \frac{1}{2}\right) \rho \leq \frac{3k}{10} - \eta$$

used in (4.3). Thus (4.2) cannot hold. This completes the proof of Lemma 9 for $k \geq 3$. \square

Proof of Lemma 9 for $k = 2$. We use the Cauchy-Schwarz inequality differently:

$$\begin{aligned} |S'|^2 &\leq \left\{ \sum_{\ell=1}^L \sum_{x \leq \frac{N}{Y}} \left| \sum_{y \sim Y} b_y e(\ell g(xy)) \right| \right\}^2 \\ &\leq \frac{LN}{Y} \sum_{y_1, y_2 \sim Y} \left| \sum_{\ell=1}^L \sum_{x \leq \frac{N}{Y}} e(\ell(g(xy_1) - g(xy_2))) \right|. \end{aligned}$$

To prove that $|S'|^2 \ll N^{2-4\eta}$ it suffices to show that

$$(4.4) \quad R := \sum_{y_1, y_2 \sim Y} \sum_{\ell=1}^L \left| \sum_{x \leq \frac{N}{Y}} e(\ell(g(xy_1) - g(xy_2))) \right| \ll N^{1-4\eta} Y L^{-1}.$$

For our proof, we need one more lemma.

Lemma 10. *Let W, X, Y be positive integers greater than 1. Then the inequality*

$$\min_{s \leq W} \left\| \frac{a(y_1^2 - y_2^2)s}{q} \right\| < \frac{1}{X}$$

is satisfied for

$$\ll \left(\frac{WY^2}{q} + 1 \right) \left(1 + \frac{q}{X} \right) (WY)^\eta$$

pairs y_1, y_2 with $y_1 \neq y_2, y_1, y_2 \sim Y$.

Proof. See [4, Lemma 7]. \square

Proof of (4.4). Since $Y \gg L^2 N^{4-\eta}$, we need only consider the contribution to R from pairs (y_1, y_2) with $y_1 \neq y_2$,

$$\sum_{\ell=1}^L \left| \sum_{x \leq \frac{N}{Y}} e(\ell(g(xy_1) - g(xy_2))) \right| > N^{1-4\eta} Y^{-1} L^{-1}.$$

For such a pair (y_1, y_2) , we apply Lemma 5, with $k = 2$; L , N/Y in place of M , X ; and $P = N^{1-4\eta}Y^{-1}L^{-1}$. We have

$$P \geq L \left(\frac{N}{Y} \right)^{1/2+\eta}$$

since $Y \ll N^{1-4\rho}$. Thus (suppressing dependence on y_1, y_2) there are natural numbers s and integers u_1, u_2 with $s \ll N^{4\rho}$, $\langle s, u_2 \rangle \leq LN^\eta$ and, for $\gamma_2 = (y_1^2 - y_2^2) \frac{a}{q}$, $\gamma_1 = (y_1 - y_2)\alpha_1$, satisfying

$$(4.5) \quad |s\gamma_j - u_j| \ll L^{-1} \left(\frac{N}{Y} \right)^{-j} N^{4\rho-2\eta} \quad (j = 1, 2).$$

It is clear that Lemma 7 is applicable. Thus

$$\begin{aligned} \sum_{\ell=1}^L \left| \sum_{x \leq \frac{N}{Y}} e(\ell(\gamma_2 x^2 + \gamma_1 x)) - s^{-1} S(s, \ell G) \int_0^{N/Y} e(\ell F(z)) dz \right| &\ll N^{2\eta} L s^{1/2} \\ &\ll N^{3\rho-5\eta} \ll \frac{N^{1-5\eta}}{Y L}. \end{aligned}$$

Thus

$$(4.6) \quad \begin{aligned} \sum_{\ell=1}^L \left| \sum_{x \leq \frac{N}{Y}} e(\ell(\gamma_2 x^2 + \gamma_1 x)) \right| &\ll \sum_{\ell=1}^L s^{-1/2} \langle s, \ell \rangle^{1/2} \frac{N}{Y} \\ &\ll L s^{-1/2} \frac{N^{1+\eta}}{Y}. \end{aligned}$$

In particular, for these pairs (y_1, y_2) we have

$$s^{1/2} \ll L^2 N^{5\eta}; \quad 2s \leq N^{4\rho}.$$

Let $X = Y^{-2} L N^{2-4\rho}$. Then (4.5) implies

$$(4.7) \quad \left\| \frac{sa(y_1^2 - y_2^2)}{q} \right\| < \frac{1}{X}.$$

The number of pairs (y_1, y_2) with $s \sim W$ (for $W \leq N^{4\rho}$) is

$$\ll \left(\frac{WY^2}{q} + 1 \right) \left(1 + \frac{q}{X} \right) N^\eta$$

by Lemma 10, and these pairs contribute to R an amount

$$\ll \frac{LN^{1+2\eta}}{YW^{1/2}} \left(\frac{WY^2}{q} + 1 \right) \left(1 + \frac{q}{X} \right)$$

(by (4.6))

$$\ll \frac{W^{1/2}LN^{1+2\eta}Y}{q} + \frac{W^{1/2}LN^{1+2\eta}Y}{X} + \frac{LN^{1+2\eta}}{Y} + \frac{LN^{1+2\eta}q}{XY}.$$

Each of these four terms is $\ll N^{1-5\eta}YL^{-1}$:

$$\frac{W^{1/2}LN^{1+2\eta}Y}{q} \ll N^{5\rho/2}Y \ll N^{1-5\eta}YL^{-1} \quad (\text{since } \rho < 2/7);$$

$$\frac{W^{1/2}LN^{1+2\eta}Y}{X} = W^{1/2}N^{-1+4\rho+2\eta}Y^3 \ll N^{-1+6\rho+2\eta}Y^3 \ll \frac{N^{1-5\eta}Y}{L}$$

(since $Y \ll N^{1-4\rho}$)

$$\frac{LN^{1+2\eta}}{Y} \ll \frac{N^{1-5\eta}Y}{L} \quad (\text{since } Y \gg N^{2\rho});$$

and

$$\frac{LN^{1+2\eta}q}{XY} = qN^{-1+4\rho+2\eta}Y \ll N^{\frac{9\rho}{2}+2\eta}Y \ll \frac{N^{1-5\eta}Y}{L}$$

(since $\rho < \frac{2}{11}$). \square

We now sum over $O(\log N)$ values of $W = N^{4\rho}2^{-j}$ and obtain the desired bound (4.4). This completes the proof of Lemma 9. \square

5. APPLICATION OF THE HARMAN SIEVE.

We use the standard notations

$$P(z) = \prod_{p < z} p,$$

$E_d = \{n : dn \in E\}$ for a finite subset E of \mathbb{N} , while

$$S(E, z) = \sum_{\substack{n \in E \\ \langle n, P(z) \rangle = 1}} 1, \quad \chi_E = \text{indicator function of } E.$$

We ‘compare’ the set A introduced in Section 1, ($A = A(f_k)$, $J = J(f_k)$) with the set

$$B = \left\{ n \in \mathbb{N} : \frac{N}{2} < n \leq N \right\}.$$

We write $[\alpha, \alpha + \beta] = [2\rho, 1 - 5\rho/2]$ ($k = 2$); $[\alpha, \alpha + \beta] = [\rho, 1 - 2J\rho]$ ($k > 2$).

Lemma 11. *Let u_h ($h \leq H$) be real numbers with $|u_h| \leq 1$, $u_k = 0$ for $(h, P(N^\eta)) > 1$. Suppose that*

$$(5.1) \quad H < MN^{-\alpha}$$

where $M \ll N^{1-5\rho/2}$ for $k = 2$ and $M \ll N^{1/2+\rho}$ for $k \geq 3$. Then, writing δ for L_1^{-1} ,

$$\sum_{h \leq H} S(A_h, N^\beta) - 2\delta \sum_{h \leq H} S(B_h, N^\beta) \ll \delta N^{1-\eta/2}.$$

Proof. We first apply [5], Lemma 1, which is a variant of [12, Theorem 3.1] convenient for our purposes. By choosing the weight function of the lemma to be

$$w(n) = \chi_A(n) - 2\delta\chi_B(n),$$

we see that we need only show that

$$\begin{aligned} \sum_{\substack{\frac{N}{2} < mn \leq N \\ m \leq M}} a_m (\chi_A(mn) - 2\delta\chi_B(mn)) &\ll \delta N^{1-2\eta/3}, \\ \sum_{N^\alpha \leq m \leq N^{\alpha+\beta}} a_m \sum_n b_n (\chi_A(mn) - 2\delta\chi_B(mn)) &\ll \delta N^{1-2\eta/3} \end{aligned}$$

where the a_m, b_n are arbitrary with $|a_m| \leq 1, |b_n| \leq 1$. Now a standard use of the upper and lower bounds for the indicator function of χ_A reduces our task to proving bounds for exponential sums that have already been given in Lemma 8 and Lemma 9; compare, for example, the arguments in [12, Section 3.4]. \square

Another ‘comparison’ that will be needed below is a bound $O(\delta N^{1-2\eta/3})$ for a sum

$$\sum_{N^\alpha \leq p \leq N^{\alpha+\beta}} (S(A_p, p) - 2\delta S(B_p, p))$$

This reduces to a sum of $O(1)$ expressions, in which $r = O(1)$, of the form

$$\sum_{\substack{p \leq a, \frac{N}{2} < pp_1 \dots p_r \leq N \\ p \leq p_1 \leq p_2 \leq \dots \leq p_r}} \{\chi_A(pp_1 \dots p_r) - 2\delta\chi_B(pp_1 \dots p_r)\}.$$

Lemma 9 provides a satisfactory estimate, using [12, Section 3.2] to remove the condition $p \leq p_1$ that will occur in the Type II exponential sums that arise.

Proof of Theorem 1 for $k \geq 3$. We observe that

$$(5.2) \quad \begin{aligned} \#\{p : p \in A\} &\geq S(A, (2N)^{1/2}) \\ &= S(A, N^\alpha) - \sum_{N^\alpha \leq p \leq (2N)^{1/2}} S(A_p, p) \end{aligned}$$

by an application of Buchstab's identity. Iterating the procedure,

$$\#\{p : p \in A\} = S_1 - S_2 - S_3 + S_4$$

where

$$\begin{aligned} S_1 &= S(A, N^\alpha), \quad S_2 = \sum_{N^\alpha \leq p \leq N^{\alpha+\beta}} S(A_p, p), \\ S_3 &= \sum_{N^{\alpha+\beta} < p \leq (2N)^{1/2}} S(A_p, N^\alpha), \\ S_4 &= \sum_{\substack{N^{\alpha+\beta} < p \leq (2N)^{1/2} \\ N^{\alpha+\beta} \leq q < p}} S(A_{pq}, q). \end{aligned}$$

Define S'_1, S'_2, S'_3 , and S'_4 in the same way as S_1, \dots, S_4 , with A replaced by B . We observe that

$$\begin{aligned} \#\{p : p \in A\} &\geq S_1 - S_2 - S_3 \\ &= 2\delta(S'_1 - S'_2 - S'_3) + O(\delta N^{1-\eta/2}) \end{aligned}$$

by two applications of Lemma 11 for S_1 and S_3 , together with one application of the remarks following Lemma 11 for S_2 . (The condition (5.1) will be trivial for S_1 , and will amount to

$$(2N)^{1/2} \leq MN^{-\rho}$$

for S_3 , where $M = 2^{1/2}N^{1/2+\rho}$.) We now follow arguments familiar from [12]. To show that $S'_1 - S'_2 - S'_3 > b(N\delta/\log N)$ for a small positive b , it suffices to show that, ω denoting the Buchstab function,

$$(5.3) \quad \iint_{0.1842 < y \ll x < 1/2, x+2y < 1} \omega\left(\frac{1-x-y}{y}\right) \frac{dx}{x} \frac{dy}{y^2} < 1.$$

(Here we use $\alpha + \beta = 2/10$ for $k = 3$ and $\alpha + \beta = 0.1842$ for $k \geq 4$.) The constant 0.1842 is close to sharp (within 10^{-4}). The validity of (5.3) was kindly checked by Andreas Weingartner. \square

Proof of Theorem 1 for $k = 2$. We now have

$$[\alpha, \alpha + \beta] = \left[\frac{4}{13}, \frac{5}{13} \right]$$

and the condition (5.1) reduces to

$$H \ll N^{8/13}.$$

We can weaken this restriction on H to

$$H \ll N^{9/13}$$

by treating the Type I exponential sums in question as Type II exponential sums with one variable between $N^{1-5/13}$ and $O(N^{1-4/13})$. Now we have stronger ‘arithmetic information’ than is used by Harman [12, Section 5.3] in the discussion of the Diophantine inequality.

$$\|p\alpha + \beta\| < p^{-0.3182} \quad \left(\text{since } \frac{2}{13} < \frac{0.3182}{2} \right).$$

We can follow the proof there verbatim to obtain Theorem 1. □

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